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Castelnuovo–Mumford regularity of simplicial toric rings

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Abstract

Bounds for the Castelnuovo–Mumford regularity of simplicial toric rings are given which are close to the bound stated in Eisenbud–Goto's Conjecture.

Keywords: Castelnuovo–Mumford regularity; Reduction number; Eisenbud–Goto's conjecture

Introduction

Let R be a standard graded algebra over a field K. Let $\mathfrak{m}$ denote the maximal homogeneous ideal of R. The Castelnuovo–Mumford regularity reg R of R is the least integer n such that $H^i_{\mathfrak{m}}(R)_{j-i} = 0$ for all $i \geqslant 0$ and all $j > n$. The interest in this invariant comes from the fact that it can be used to bound the degrees of generators of syzygy modules of R (see [3]) or the degree of certain Gröbner bases of the defining ideal of R. One of long-standing open problem is to prove the following:

Eisenbud–Goto's Conjecture. If R is a domain over an algebraically closed field K then

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$$\operatorname{reg} R \leqslant \deg R - \operatorname{codim} R,$$

where $\deg R$ denotes the multiplicity of R with respect to $\mathfrak{m}$ and $\operatorname{codim} R = \dim_K [R]_1 - \dim R$.

This conjecture was proved for several special cases:

(1) for 2-dimensional case by Gruson, Lazarsfeld and Peskine [4],
(2) for 3- and 4-dimensional cases by Lazarsfeld [8] and Ran [9] provided that the projective scheme $\operatorname{Proj}(R)$ is smooth,
(3) for the case $\deg R \leqslant \operatorname{codim} R + 2$ by Hoa, Stückrad and Vogel [5],
(4) for Cohen–Macaulay or Buchsbaum rings by Treger [17] and Stückrad and Vogel [14].

This conjecture is widely open in general, even for toric rings, i.e., semigroup rings of homogeneous affine semigroups. In this case Peevaa and Sturmfels [10] could prove the conjecture for rings in codimension 2. For a toric ring in a higher codimension, Sturmfels [16] got the following bound: $\operatorname{reg} R \leqslant (\dim R + \operatorname{codim} R) \operatorname{codim} R \deg(R)$. This is a very good bound, because it is a linear function in terms of the multiplicity.

The aim of this paper is to show that if we restrict to the subclass of simplicial toric rings, then the Castelnuovo–Mumford regularity can be bounded by a function which is not very far from the one given in Eisenbud–Goto's Conjecture. Namely we get the following main result (see Theorem 3.5):

Theorem. *Let S be a simplicial affine semigroup and $K[S]$ the semigroup ring of S. Assume that* $\deg K[S] \geqslant \operatorname{codim} K[S] + 2$. *Then*

$$\operatorname{reg} K[S] \leqslant \min\bigl\{\operatorname{codim} K[S]\bigl(\deg K[S] - 1\bigr), \dim K[S]\bigl(\deg K[S] - \operatorname{codim} K[S] - 2\bigr) + 2\bigr\}.$$

In order to prove this theorem we first establish a linear relation between the Castelnuovo–Mumford regularity and certain reduction number $r(S)$ of $K[S]$ (Theorems 3.1 and 3.2). To do that we use two approaches. One is to compute the local cohomology modules of $K[S]$ via the local cohomology modules of certain monomial ideals, and the other is to use the formula for computing the local cohomology modules of $K[S]$ given in [11]. In both approaches it turns out that the assumption S being simplicial is very essential. With these methods we can get bounds for $\operatorname{reg} K[S]$ which are in many cases even much smaller than $\deg K[S] - \operatorname{codim} K[S]$. Further we show that the reduction number $r(S)$ can be bounded exactly by the bound given in the above conjecture (Theorem 1.1). After that the main result immediately follows. As a consequence we can show that for many simplicial toric rings Eisenbud–Goto's conjecture is true (Corollary 3.6 and Proposition 3.7).

The paper is organized as follows. In Section 1 we establish several bounds for the reduction number $r(S)$. In Section 2 we first recall the result of [11], and then give a decomposition of $K[S]$ into a direct sum of certain submodules. From that we can compute the local cohomology modules of $K[S]$ via that of certain monomial ideals. In the last section we establish various bounds for the Castelnuovo–Mumford regularity.

1. Bounds for reduction numbers

Let S be a homogeneous, simplicial affine semigroup. That means S is a semigroup generated by a set of elements of the following type:

$$\mathcal{A} = \{e_1, \ldots, e_d, a_1, \ldots, a_c\} \subseteq \mathbf{N}^d,$$

where

$$e_1 = (\alpha, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, \alpha),$$
$$a_1 = (a_{i[1]}, \ldots, a_{i[d]}) \quad \text{with } a_{i[1]} + \cdots + a_{i[d]} = \alpha,\ i = 1, \ldots, c.$$

Moreover we can assume that the integers $a_{i[j]}$, $i = 1, \ldots, c$, $j = 1, \ldots, d$, are relatively prime. Since the semigroup ring $K[S]$ is isomorphic to a polynomial ring if $c = 0$, we will consider only the case $c \geqslant 1$ and $\alpha \geqslant 2$. Note that $d = \operatorname{rank}(\mathbf{Z}(S))$ and $c = \operatorname{codim} K[S]$. For an element u of the group $\mathbf{Z}(S)$ generated by S, the degree of u is defined as $\deg u = (u_{[1]} + \cdots + u_{[d]})/\alpha$.

Let $A = A_0 \oplus A_1 \oplus \cdots$, where $A_0 = K$, be a standard graded K-algebra of dimension d. A minimal reduction of A is a graded ideal I generated by d linear forms such that $[IA]_n = A_n$ for $n \gg 0$. The least integer n such that $[IA]_{n+1} = A_{n+1}$ is called the reduction number of A w.r.t. I and will be denoted by $r_I(A)$. Vasconcelos [21] showed that if $\operatorname{chac}(K) = 0$ and A is a domain, then $r_I(A) \leqslant \deg A - 1$, where $\deg A$ is the multiplicity of A w.r.t. the maximal graded ideal of A. On the other hand it is known that $r_I(A) \leqslant \operatorname{reg} A$ (see [18]). Since Eisenbud–Goto's Conjecture is still open, it would be nice to solve the following problem:

Problem. If A is a domain, then $r_I(A) \leqslant \deg A - \operatorname{codim}(A)$.

As usual we can identify the affine semigroup ring $K[S]$ with the subring of the polynomial ring $k[t_1, \ldots, t_d]$ generated by monomials $t^u := t_1^{u_{[1]}} \cdots t_d^{u_{[d]}}$, where $u \in S$. Note that $(t_1^\alpha, \ldots, t_d^\alpha)$ is a minimal reduction of $K[S]$. We denote by $r(S)$ the reduction number of $K[S]$ w.r.t. this minimal reduction. Then $r(S)$ is the least positive integer r such that $(r+1)\mathcal{A} = \{e_1, \ldots, e_d\} + r\mathcal{A}$, where for two subsets B and C of $\mathbf{Z}^d$ we denote by $B \pm C$ the set of all elements of the form $b \pm c$, $b \in B$, $c \in C$, and $nB = B + \cdots + B$ (n times). It is clear that $r(S)$ does not depend on K. Hence, by Vasconcelos' result, $r(S) \leqslant \deg K[S] - 1$ for all K. However for $r(S)$ we can completely solve the above problem:

Theorem 1.1. $r(S) \leqslant \deg K[S] - \operatorname{codim} K[S]$.

Proof. We introduce the following equivalence relation for the elements of $\mathbf{Z}(S)$: $u \sim v$ if $u - v \in \alpha\mathbf{Z}^d$. Let $D \subset \mathbf{Q}^d$ denote the parallelepiped:

$$D = \left\{x \in \mathbf{Q}^d;\ 0 \leqslant x_{[i]} < \alpha, \forall i\right\}.$$

It is clear that each point of $\mathbf{Z}(S)$ is equivalent to a point in $\mathbf{Z}(S) \cap D$ (see also [13, Lemma 4.6.7]). Note that we can embed the group $\mathbf{Z}(S)$ into an Euclidean space of dimension d such that $\mathbf{Z}(S)$ is the lattice of all integral points. Since $e := \deg K[S]$ is equal to the Euclidean volume of D in this space, by [2], Theorem 5.1 and Corollary 5.2 (see also [13, Proposition 4.6.30]) it follows that $e = \sharp(\mathbf{Z}(S) \cap D)$. Thus there are exactly e equivalence classes.

Let m_k be nonnegative integers with $m_1 + \cdots + m_c = e - c + 1$. We need to show that

$$\sum_{k=1}^{c} m_k a_k \in \{e_1, \ldots, e_d\} + S. \tag{1}$$

Consider the following non-zero partial sums of $\sum_{k=1}^{c} m_k a_k$:

$$\begin{aligned} &b_1 = a_1,\ \ldots,\ b_{m_1} = m_1 a_1,\ b_{m_1+1} = m_1 a_1 + a_2,\ \ldots, \\ &b_{e-c+1} = m_1 a_1 + \cdots + m_c a_c. \end{aligned}$$

If there is $2 \leqslant k \leqslant e - c + 1$ such that $b_k \sim a_l$ for some $0 \leqslant l \leqslant c$, where $a_0 := 0$, then $b_k - a_l \in \alpha\mathbf{Z}^d$. Since $0 \leqslant a_{l[i]} < \alpha$, $b_{k[i]} \geqslant 0$ for all i and $\deg b_k \geqslant 2$, it follows that $b_k - a_l = p_1 e_1 + \cdots + p_d e_d$ for some nonnegative p_i with $p_1 + \cdots + p_d > 0$. Since b_k is a partial sum of $\sum_{k=1}^{c} m_k a_k$, (1) follows. If non of these $(e - c)$ elements is equivalent to an element a_i, $0 \leqslant i \leqslant c$, then we can find $k' < l'$ such that $b_{k'} \sim b_{l'}$. Then $b_{l'} - b_{k'} = p'_1 e_1 + \cdots + p'_d e_d$ for some nonnegative p'_i with $p'_1 + \cdots + p'_d > 0$. Hence we also get (1). $\square$

Under certain additional assumptions on S we can get other bounds for $r(S)$ which in many cases are better than the above one. Let us introduce some notation.

Let $\mathcal{P}$ denote the convex polytope spanned by $\mathcal{A}$. Note that $\mathcal{P}$ is a $(d-1)$-dimensional polytope whose faces are spanned by

$$\mathcal{P}_I = \{x \in \mathcal{A};\ x_{[i]} = 0 \text{ for all } i \in I\},$$

where $I \subseteq \{1, \ldots, d\}$. Let $\overline{\mathcal{P}}_I$ denote the corresponding face of $\mathcal{P}$. Let

$$M_{\alpha,d} = \left\{u \in \mathbf{N}^d;\ u_{[1]} + \cdots + u_{[d]} = \alpha\right\}.$$

We say that a face $\overline{\mathcal{P}}_I$ is *full* if it contains all points of $M_{\alpha,d}$ lying on this face, i.e., if $\mathcal{P}_I = \overline{\mathcal{P}}_I \cap M_{\alpha,d}$. Finally, let

$$F_I = \langle \mathcal{P}_I \rangle = \{x \in S;\ x_{[i]} = 0 \text{ for all } i \in I\}.$$

In particular $F_\emptyset = S$ and $F_{\{i,\dots,d\}} = \{0\}$. For simplicity we also write $\mathcal{P}_i$, F_i instead of $\mathcal{P}_{\{i\}}$, $F_{\{i\}}$.

Recall that S is called a *normal semigroup* if $S = \mathbf{Q}_+ S \cap \mathbf{Z}(S)$, where $\mathbf{Q}_+ S$ denotes the set of all linear combinations of elements of S with nonnegative rational coefficients. It is clear that if a face $\overline{\mathcal{P}}_I$ is full, then F_I is a normal semigroup. We have

Lemma 1.2. *Assume that $\mathcal{P}$ has a full face of dimension i. Then*

$$r(S) \leqslant \alpha^{d-1-i} + i - 1.$$

Proof. We use induction $d-1-i$. If $i = d-1$ then $\mathcal{A} = M_{\alpha,d}$ and S is a normal affine semigroup ring. In this case $K[S]$ is a Cohen–Macaulay ring and $\operatorname{reg} K[S] \leqslant d-1$. The first fact was proven in [7] and the later one was proven in [15], Theorem 13.14 or can be deduced from the proof of [20], Corollary 4.7. Since in this case $r(S) = \operatorname{reg} K[S]$, we have $r(S) \leqslant d-1 = i$.

Let $d-1 > i$ and $\alpha \geqslant 2$. W.l.o.g. we may assume that the face $\overline{\mathcal{P}}_I$ is full, where $I = \{1,\dots,d-i-1\}$, $a_1,\dots,a_p \notin \mathcal{P}_I$ and $a_{p+1},\dots,a_c \in \mathcal{P}_I$. Let $m_1,\dots,m_c$ be nonnegative integers with $m_1 + \cdots + m_c = \alpha^{d-1-i} + i$. We need to show that

$$\sum_{k=1}^{c} m_k a_k \in \{e_1,\dots,e_d\} + S. \tag{2}$$

Assume $m_{p+1} + \cdots + m_c \geqslant i+1$. Since the simplicial subsemigroup F_I (of dimension $i+1$) is a normal affine semigroup and $a_{p+1},\dots,a_c \in F_I$, from the first step of induction we have $r(F_I) \leqslant i$. Hence

$$m_{p+1}a_{p+1} + \cdots + m_c a_c \in \{e_{d-i},\dots,e_d\} + F_I,$$

which implies (2).

Let $m_{p+1} + \cdots + m_c \leqslant i$. Then $m_1 + \cdots + m_p \geqslant \alpha^{d-1-i}$. Consider the following $m_1 + \cdots + m_p$ elements:

(∗) $b_{1,1} = a_1,\ \dots,\ b_{1,m_1} = m_1 a_1,$
$b_{1,m_1+1} = m_1 a_1 + a_2,\ \dots,\ b_{1,m_1+m_2} = m_1 a_1 + m_2 a_2, \dots,$
$b_{1,m_1+\cdots+m_p} = m_1 a_1 + \cdots + m_p a_p.$

These elements are non-zero partial sums of the sum $m_1 a_1 + \cdots + m_p a_p$ and have the following property:

(∗∗) For all $i < j$ and all $1 \leqslant k \leqslant p$ we have $0 \leqslant n_{1ik} \leqslant n_{1jk} \leqslant m_k$,
where n_{1ik}, n_{1jk} are coefficients of a_k in $b_{1,i}$ and $b_{1,j}$, respectively.

There are two cases:

(i) There are $s_1 = \alpha^{d-2-i}$ elements, say $b_{1,j_1}, \dots, b_{1,j_{s_1}}$ $(j_1 < \cdots < j_{s_1})$, whose first coordinates are divisible by α. Then we set

$$b_{2,1} = b_{1,j_1}, \ \dots, \ b_{2,s_1} = b_{1,j_{s_1}}.$$

(ii) There are at most $s_1 - 1$ elements satisfying the property in (i). Then there are at least $s_1(\alpha - 1) + 1$ elements among $b_{1,j}$ whose first coordinates are not divisible by α. Then one can find $s_1 + 1$ elements, say $b_{1,j_1}, \dots, b_{1,j_{s_1+1}}$ $(j_1 < \cdots < j_{s_1+1})$, whose first coordinates are in the same residue class modulo α. Set

$$b_{2,1} = b_{1,j_2} - b_{1,j_1}, \ \dots, \ b_{2,s_1} = b_{1,j_{s_1+1}} - b_{1,j_1}.$$

Thus in both cases we can find s_1 elements $b_{2,1}, \dots, b_{2,s_1}$ whose first coordinates are divisible by α. Moreover these elements are non-zero partial sums of the sum $m_1a_1 + \cdots + m_pa_p$ and satisfy property $(**)$ (with replacement 1 by 2 in the indices). So we can repeat this process with the second coordinate, and so on. After $(d - i - 1)$ steps we can find a non-zero partial sum $b = n_1a_1 + \cdots + n_pa_p$ of $m_1a_1 + \cdots + m_pa_p$ such that all first $(d - i - 1)$ coordinates of b are divisible by α. Let

$$b' = b - (b_{[1]}e_1 + \cdots + b_{[d-1-i]}e_{d-1-i}) \in \mathbf{Q}_+F_I \cap \mathbf{Z}(S).$$

Since $\overline{\mathcal{P}}_I$ is full, F_I is a normal semigroup. Hence $b' \in F_I \subseteq S$. On the other hand $a_1, \dots, a_p \notin \mathcal{P}_I$. Therefore $b_{[1]} + \cdots + b_{[d-1-i]} > 0$. From this it follows that $b \in \{e_1, \dots, e_{d-1-i}\} + S$, which yields (2). □

If $\mathcal{P}$ has no full face, but one of its faces contains a lot of points of $\mathcal{A}$, then $r(S)$ may be estimated as follows:

Lemma 1.3. *Assume that a p-dimensional face $\overline{\mathcal{P}}_I$ contains $q + p + 1$ points of $\mathcal{A}$, where $p \leqslant d - 1$. Then*

$$r(S) \leqslant (\alpha^p - q)\alpha^{d-1-p}.$$

Proof. Let $m_1, \dots, m_c$ be nonnegative integers with $m_1 + \cdots + m_c = \beta\alpha^{d-1-p} + 1$, where $\beta = \alpha^p - q$. We need to show that

$$\sum_{k=1}^{c} m_ka_k \in \{e_1, \dots, e_d\} + S.$$

The proof uses the same idea as in the proof of the previous lemma. We may assume that $I = \{1, \dots, d - p - 1\}$ and the points of $\mathcal{P}_I \setminus \{e_1, \dots, e_d\}$ are enumerated as $a_{c-q+1}, \dots, a_c$. Set $t = m_1 + \cdots + m_{c-q}$ and $u = m_{c-q+1} + \cdots + m_c$. If $u = 0$ then, as in the proof of the previous lemma, one can define β non-zero partial sums $b_1, \dots, b_\beta$ of $\sum_{k=1}^{c-q} m_ka_k$ whose first $(d - 1 - p)$ coordinates are divisible by α. Let $u > 0$. Since $\deg K[F_I] \leqslant \alpha^p$, by Theorem 1.1 we may assume that $u \leqslant \beta$. We have $t = \beta\alpha^{d-1-p} + 1 - u \geqslant (\beta - u + 1)\alpha^{d-1-p}$ and $\beta - u + 1 \geqslant 1$. Hence we can define $(\beta - u + 1)$ non-zero partial sums $b_1, \dots, b_{\beta-u+1}$ of $\sum_{k=1}^{c-q} m_k a_k$ whose first $(d - 1 - p)$ coordinates are divisible by α. Let $b_{\beta-u+i} = b_{\beta-u+1} + b'_i$, $i = 2, \dots, u$, where $b'_1, \dots, b'_s$ are the partial sums $a_{c-q+1}, \dots, m_{c-q+1}a_{c-q+1}, \dots, m_{c-q+1}a_{c-q+1} + \dots + m_c a_c$. So in both cases $b_1, \dots, b_\beta$ have been defined and these non-zero partial sums $\sum_{k=1}^{c} m_k a_k$ satisfy $(**)$ (with obvious modification). Now applying the same argument as in the proof of Theorem 1.1 to the vectors consisting of the last $(p + 1)$ coordinates of $b_1, \dots, b_\beta, a_{c-q+1}, \dots, a_c$, we get a non-zero partial sum b of $\sum_{k=1}^{c} m_k a_k$ such that all its coordinates are divisible by α. From this it follows that $\sum_{k=1}^{c} m_k a_k \in \{e_1, \dots, e_d\} + S$. $\square$

Example 1.4. Note that we always have $\deg K[S] \leqslant \alpha^{d-1}$. Very often we have equality here. This is the case, if $\mathcal{A}$ contains the following d elements $(u_{[1]}, \dots, u_{[d]})$, $(u_{[1]} - 1, u_{[2]} + 1, u_{[3]}, \dots, u_{[d]})$, …, $(u_{[1]} - 1, u_{[2]}, \dots, u_{[d-2]}, u_{[d-1]} + 1, u_{[d]})$, where $u_{[1]}, \dots, u_{[d]}$ are nonnegative integers such that $u_{[1]} + \dots + u_{[d]} = \alpha$ and $0 < u_{[1]} < d$. For such a semigroup S the bound in Theorem 1.1 is $\alpha^{d-1} - c$. If, in addition, the condition of Lemma 1.3 holds then $(\alpha^p - q)\alpha^{d-1-p} \ll \alpha^{d-1} - c$ if $c \ll q\alpha^{d-1-p}$. Assume now that the condition of Lemma 1.2 is satisfied with $i > 0$. If d is fixed then $\alpha^{d-1} - c > \alpha^{d-1} - \binom{\alpha+d-1}{d-1}$, which is a polynomial of α of degree $d - 1$, while the bound in Lemma 1.2 is a polynomial of degree $d - 1 - i$.

2. On the structure of local cohomology modules

The local cohomology of affine semigroup rings were described in [11] and [20]. First, let us recall the main formula for computing the local cohomology modules given in [11].

Let $\mathcal{C}$ denote the convex polyhedral cone spanned by S in $\mathbf{Q}^d$. Since S is the simplicial affine semigroup, $\mathcal{C}$ is exactly the quadrant $\mathbf{Q}_+^d$. The set $\mathcal{F}_S$ of all its faces is $\mathbf{Q}_+ F_I$; $I \subseteq \{1, \dots, d\}$. Thus there is a one-to-one inclusion reversing correspondence between the lattice $\mathcal{F}_S$ and the full simplicial complex Δ on the vertex set $\{1, \dots, d\}$. Note that the empty set of Δ corresponds to the largest face $\mathcal{C}$, and $\{1, \dots, d\}$ corresponds to the zero-dimensional face $\{0\}$ of $\mathcal{C}$.

By a *subcomplex* π of Δ we mean a simplicial complex whose vertex set is a subset of $\{1, \dots, d\}$ (note that π should always contain the empty set $\emptyset$). We will use the notation $\pi < \Delta$ to say that π is a proper subcomplex of Δ. For a fixed subcomplex π of Δ we define

$$S_\pi = \bigcap_{I \in \pi} (S - F_I) \Big\backslash \bigcup_{I \notin \pi} (S - F_I).$$

Let $\mathfrak{m} = K[S \setminus \{0\}]$ denote the maximal homogeneous ideal of the semigroup ring $K[S]$. Then the local cohomology of $K[S]$ can be computed as follows:

Lemma 2.1. *For all $i \geqslant 0$ there is an isomorphism of $\mathbf{Z}^d$-graded K-vector spaces*

$$H^i_{\mathfrak{m}}\big(K[S]\big) \cong \bigoplus_{\pi < \Delta} K[S_\pi] \otimes \widetilde{H}_{d-i-1}(\pi, K).$$

Proof. Let $\mathcal{U} = \{\mathbf{Q}_+ F_I;\ I \in \pi\}$. Since S is a simplicial semigroup, $\mathcal{U}$ is a filter of $\mathcal{F}_S$, i.e., if $G \in \mathcal{U}$ and $F \in \mathcal{F}_S$ such that $G \subset F$, then $F \in \mathcal{U}$. According to [11] let

$$S_{\mathcal{U}} = \bigcap_{\mathbf{Q}_+ F_I \in \mathcal{U}} (S - F_I) \Big\backslash \bigcup_{\mathbf{Q}_+ F_I \notin \mathcal{U}} (S - F_I),$$

and

$$\Delta_{\mathcal{U}} = \big\{\{i_1, \dots, i_t\} \subseteq \{1, \dots, d\};\ \mathbf{Q}_+ F_{(\{1,\dots,d\}\setminus\{i_1\})\cap\cdots\cap(\{1,\dots,d\}\setminus\{i_t\})} \notin \mathcal{U}\big\}.$$

By [11], Corollaries 2.2 and 5.1 we have

$$H^i_{\mathfrak{m}}\big(K[S]\big) \cong \bigoplus_{\mathcal{U} \subseteq \mathcal{F}_S} K[S_{\mathcal{U}}] \otimes_K \widetilde{H}_{i-2}(\Delta_{\mathcal{U}}, K),$$

where the sum is taken over all filters $\mathcal{U}$ of $\mathcal{F}_S$. Obviously $S_{\mathcal{U}} = S_\pi$ and

$$\Delta_{\mathcal{U}} = \big\{\{i_1, \dots, i_t\} \subseteq \{1, \dots, d\};\ \{1, \dots, d\} \setminus \{i_1, \dots, i_t\} \notin \pi\big\}.$$

On the other hand, by duality we have $\widetilde{H}_{i-2}(\Delta_{\mathcal{U}}, K) \cong \widetilde{H}_{d-i-1}(\pi, K)$. Hence

$$H^i_{\mathfrak{m}}\big(K[S]\big) \cong \bigoplus_{\pi < \Delta} K[S_\pi] \otimes \widetilde{H}_{d-i-1}(\pi, K). \qquad \square$$

In the above formula we have a decomposition of $H^i_{\mathfrak{m}}(K[S])$ into a direct sum of certain vector subspaces. In the sequel we give another approach which enables us to get a decomposition of $H^i_{\mathfrak{m}}(K[S])$ into a direct sum of certain submodules (of course, only for simplicial semigroup rings). We need the following notation:

$$H = \langle a_1, \dots, a_c\rangle, \qquad T = K\big[t_1^\alpha, \dots, t_d^\alpha\big] \cong K[y_1, \dots, y_d].$$

It is clear that $K[S] = T[H]$. Since $(t^u)^\alpha \in T$ for all $u \in H$, $K[S] = T[H]$ is a finite module over T and therefore there is a uniquely determined finite set $\{b_1, \dots, b_\mu\} \subset H$ such that $K[S] = (m_1, \dots, m_\mu)T$ with $m_i := t^{b_i}$, $i = 1, \dots, \mu$. Let $m_i \sim m_j$ if and only if $b_i - b_j \in \alpha\mathbf{Z}^d$ (i.e., $b_i \sim b_j$ according to the definition in the proof of Theorem 1.1).

This is an equivalence relation on $\{m_1, \dots, m_\mu\}$. For any element $u \in \mathbf{Z}(S)$ we can find $u_1, u_2 \in H$ and $u_3 \in \alpha\mathbf{Z}^d$ such that $u = u_1 - u_2 + u_3$. Since $u = u_1 + (\alpha - 1)u_2 - \alpha u_2 + u_3 \sim u_1 + (\alpha - 1)u_2 \in H$, one can find an index i, $1 \leqslant i \leqslant \mu$, such that $u \sim b_i$. Thus the number of equivalence classes of $\mathbf{Z}(S)$ is equal to that of $b_1, \dots, b_\mu$. From the proof of Theorem 1.1 it follows that $m_1, \dots, m_\mu$ are divided exactly into $e := \deg K[S]$ equivalence classes. Let $\beta_1 = \{1\}, \beta_2, \dots, \beta_e$ denote the corresponding classes. We define for $i = 1, \dots, e$ the monomial $n_i = \gcd\{m;\ m \in \beta_i\}$ and set $n_i = t^{h_i}$ with $h_i \in \mathbf{N}^d$. Let

$$\tilde{\beta}_i = \{m/n_i;\ m \in \beta_i\} \subset T \quad \text{and} \quad I_i = \tilde{\beta}_i T.$$

$I_1, \dots, I_e$ are monomial ideals in T. Since $\gcd(m;\ m \in I_i,\ m \text{ monomial}) = 1$ by construction, we have $\dim T/I_i \leqslant d-2$ for all $i = 1, \dots, e$. Note that all modules under consideration are $\mathbf{Z}^d$-graded if we take the degree of elements of S for that of the corresponding monomials. Denote by $\mathfrak{m}_T$ the maximal graded ideals of T. The following result says that one can compute the local cohomology modules of $K[S]$ via the computation of that of certain monomial ideals.

Proposition 2.2. *There are isomorphisms of $\mathbf{Z}^d$-graded T-modules*:

(i) $K[S] \cong \bigoplus_{j=1}^{e} I_j(-h_j)$,
(ii) $H^i_{\mathfrak{m}}(K[S]) \cong \bigoplus_{j=1}^{e} H^i_{\mathfrak{m}_T}(I_j)(-h_j)$ *for all* $i \geqslant 0$.

Proof. (i) Define

$$\eta : \bigoplus_{j=1}^{e} I_j(-h_j) \to K[S],$$

by $\eta(f_1, \dots, f_e) = \sum_{j=1}^{e} f_j n_j$. It is clear that η is a T-epimorphism preserving the $\mathbf{Z}^d$-grading. Let $\tau \in \operatorname{Ker}\eta$. W.l.o.g. assume that τ is homogeneous, i.e., $\tau = (\alpha_1 m'_1, \dots, \alpha_e m'_e)$ with $\alpha_1, \dots, \alpha_e \in K$ and monomials $m'_j \in I_j$, $j = 1, \dots, e$. Let $m'_j = t^{c_j}$, $c_j \in \langle e_1, \dots, e_d \rangle$, $j = 1, \dots, e$. Since $c_p + h_p \neq c_q + h_q$ for $p, q \in \{1, \dots, e\}$, $p \neq q$ (we even have $(c_p + h_p) - (c_q + h_q) \equiv h_p - h_q \not\equiv 0 \bmod (\alpha \mathbf{Z}^d)$), there is some $l \in \{1, \dots, e\}$ with $\alpha_j = 0$ for all $j \neq l$, i.e., $\tau = (0, \dots, 0, \alpha_l m'_l, 0, \dots, 0)$. Then $\alpha_l m'_l m_l = 0$, and hence $\alpha_l = 0$, i.e., $\tau = 0$. Therefore η is injective and hence an isomorphism.

(ii) The second statement follows from (i), since $H^i_{\mathfrak{m}}(K[S]) \cong H^i_{\mathfrak{m}_T}(K[S])$ as T-modules (see, e.g., [1, Theorem 13.1.6]). □

Remark. By the definition of the reduction number $r(S)$ it follows that

$$\max\{\deg b_1, \dots, \deg b_\mu\} = r(S).$$

In order to calculate I_j we can consider all elements of S of degree at most $r(S)$ (or $\alpha^{d-1} - c$ if $r(S)$ is not known) instead of the minimal basis $\{b_1, \dots, b_\mu\}$, which usually needs more computation.

For a $\mathbf{Z}$-graded module M we set

$$a(M) = \begin{cases} \max\{n;\ M_n \neq 0\} & \text{if } M \neq 0, \\ -\infty & \text{if } M = 0. \end{cases}$$

Thus if M is an artinian module, then $a(M) < \infty$. Note that $a(H^d_{\mathfrak{m}}(K[S]))$ is often called the a-invariant of $K[S]$. We can easily compute this invariant as follows:

Corollary 2.3.

$$H^d_{\mathfrak{m}}\big(K[S]\big) \cong \bigoplus_{j=1}^{e} T^\vee\big(d^* - h_j\big),$$

where $T^\vee$ denotes the Matlis dual of T, i.e., $T^\vee \cong K[y_1^{-1}, \dots, y_d^{-1}]$, and $d^* = (\alpha, \dots, \alpha)$. Especially

$$a\big(H^d_{\mathfrak{m}}\big(K[S]\big)\big) = \max\{\deg h_j;\ 1 \leqslant j \leqslant e\} - d.$$

Proof. By construction we have $\dim T/I_j \leqslant d-2$ for all $j = 1, \dots, e$. Therefore from the exact sequence

$$0 \to I_j \to T \to T/I_j \to 0,$$

and Proposition 2.2 we get

$$H^d_{\mathfrak{m}_T}(I_j) \cong H^d_{\mathfrak{m}_T}(T).$$

Since T is Gorenstein, $H^d_{\mathfrak{m}_T}(T) \cong T^\vee(d^*)$. Hence the isomorphism follows from Proposition 2.2(ii). In particular, we have

$$\begin{aligned} d + a\big(H^d_{\mathfrak{m}}\big(K[S]\big)\big) &= \max\big\{d + a\big(T^\vee(d^*)\big) + \deg h_j;\ 1 \leqslant j \leqslant e\big\} \\ &= \max\{\deg h_j;\ 1 \leqslant j \leqslant e\}. \qquad \square \end{aligned}$$

Corollary 2.4. *$K[S]$ is a Cohen–Macaulay ring if and only if $I_j = T$ for all $j = 1, \dots, e$.*

Proof. By Proposition 2.2(ii) we have

$$H^i_{\mathfrak{m}}\big(K[S]\big) \cong \bigoplus_{j=0}^{e} H^i_{\mathfrak{m}_T}(I_j)(-h_j) \cong \bigoplus_{j=0}^{e} H^{i-1}_{\mathfrak{m}_T}(T/I_j)(-h_j)$$

for all $i < d$. Hence $K[S]$ is Cohen–Macaulay if and only if $H^i_{\mathfrak{m}_T}(T/I_j) = 0$ for all $i \leqslant d-2$ and all $j = 1, \dots, e$. Since $\dim T/I_j \leqslant d-2$, this is possible if and only if $I_j = T$ for all $j = 1, \dots, e$. $\square$

3. Bounds for Castelnuovo–Mumford regularity

In this section, using the results of previous sections we can give various bounds for the Castelnuovo–Mumford regularity. In particular we are able to show that for many affine semigroup rings the Eisenbud–Goto's conjecture is true. Recall that the Castelnuovo–Mumford regularity can be defined as follows:

$$\operatorname{reg} K[S] = \max\bigl\{i + a\bigl(H^i_{\mathfrak{m}}\bigl(K[S]\bigr)\bigr);\ i = 0, \dots, d\bigr\}.$$

Theorem 3.1. *Let $r(S)$ be the reduction number of $K[S]$ w.r.t. $(t_1^\alpha, \dots, t_d^\alpha)$.*

(i) *If $r(S) \leqslant 1$, then* $\operatorname{reg} K[S] \leqslant 1$.
(ii) *If $r(S) > 1$, then* $\operatorname{reg} K[S] \leqslant d(r(S) - 2) + 2$.

Proof. By Proposition 2.2(ii)

$$\operatorname{reg} K[S] \leqslant \max\{\operatorname{reg} I_j + \deg h_j;\ j = 1, \dots, e\}.$$

Fix an index j. Denote by d_j the maximal degree of elements in a minimal generating set of I_j. Note that $d_j = 0$ if $I_j = T$. From the construction of I_j it follows that

$$\max\{d_j + \deg h_j;\ j = 1, \dots, e\} = \max\{\deg b_1, \dots, \deg b_\mu\} = r,$$

where $r = r(S)$. Hence $d_j \leqslant r - \deg h_j$ and $\deg h_j \leqslant r$. There are two cases:

If $I_j = T$, then

$$\operatorname{reg} I_j + \deg h_j = \deg h_j \leqslant r.$$

Assume that $I_j \neq T$. Then $j > 1$, $h_j \neq 0$, and so $\deg h_j \geqslant 1$. Since I_j is a monomial ideal in d variables, by [6], Theorem 3.4 we have

$$\operatorname{reg} I_j \leqslant 1 + d(d_j - 1) \leqslant 1 + d(r - \deg h_j - 1).$$

Hence

$$\begin{aligned}\operatorname{reg} I_j + \deg h_j &\leqslant 1 + \deg h_j + d(r - \deg h_j - 1) = 1 + d(r-1) - (d-1)\deg h_j \\ &\leqslant 1 + d(r-1) - (d-1) = 2 + d(r-2).\end{aligned}$$

Let j run through the set $\{1, \dots, e\}$ we get

$$\operatorname{reg} K[S] \leqslant \max\{\operatorname{reg} I_j + \deg h_j;\ j = 0, \dots, e\} \leqslant \max\bigl\{r, d(r-2) + 2\bigr\},$$

which implies (i) and (ii) of the theorem. □

In view of the above theorem and Theorem 1.1 it would be nice to know how big the difference $\operatorname{reg} K[S] - r(S)$ could be. In all examples we have calculated this difference is zero.

For a number x we denote by $\lceil x \rceil$ the least integer not less than x. The following bounds are in many cases smaller than the bound given in the previous theorem and the bound $\deg K[S] - \operatorname{codim} K[S]$.

Theorem 3.2. *With the above notations we have*

(i) $\operatorname{reg} K[S] \leqslant c(\alpha - 1)$,
(ii) $\operatorname{reg} K[S] \leqslant d \cdot r(S) - \lceil \frac{d \cdot r(S)}{\alpha} \rceil \leqslant d \cdot r(S) - 1$.

Proof. Assume that for a fixed simplicial complex π with $\widetilde{H}_{d-p-1}(\pi, K) \neq 0$, the subset S_π is not empty. Since $H^p_{\mathfrak{m}}(K[S])$ is an artinian module over $K[S]$, from Lemma 2.1 it follows that elements of S_π must have bounded degrees. Assume that $x \in S_\pi$ is an element of the maximal degree. Fix an index i. For any $J \notin \pi$ we have $x \in S - F_j$. Hence $x + e_i \in S - F_j$ too. However, by the assumption, $x + e_i \notin S_\pi$. By the definition of S_π this implies that there should be a subset $I \in \pi$ such that $x + e_i \in S - F_j$. Fix this I. Then we can find an element $u_I \in F_I$ and non-negative integers m_h, n_l such that

$$x + e_i + u_I = \sum_{h=1}^{d} m_h e_h + \sum_{l=1} n_l a_l. \tag{3}$$

Since $x \notin S - F_I$, $x + u_I \notin S$. Hence in the right hand of the above relation e_i cannot appear and we should have $m_i = 0$.

(i) Note that if $x \in \mathcal{A}$ such that $x \notin F_i$, then $x_{[i]} > 0$ and we have $\alpha x = x_{[1]}e_1 + \cdots + x_{[d]}e_d$. So, if in (3) $n_l \geqslant \alpha$ for some $a_l \notin F_i$ it would imply $x + u_I \in S$, a contradiction. Hence we must have $n_l \leqslant \alpha - 1$ for all such indices. Rewriting (3) as follows:

$$x + e_i + u_I = \sum_{h=1}^{d} m_h e_h + \sum_{a_l \notin F_i} n_l a_l + \sum_{a_l \in F_i} n_l a_l,$$

we get

$$x_{[i]} + \alpha \leqslant \sum_{a_l \notin F_i} n_l a_{l[i]} \leqslant (\alpha - 1) \sum_{a_l \notin F_i} a_{l[i]} = (\alpha - 1) \sum_{l=1}^{c} a_{l[i]}.$$

Hence

$$\sum_{i=1}^{d} x_{[i]} + d\alpha \leqslant (\alpha - 1) \sum_{l=1}^{c} \sum_{i=1}^{d} a_{l[i]} = c(\alpha - 1)\alpha.$$

So $\deg x \leqslant c(\alpha - 1) - d$. Hence, by Lemma 2.1, $a(H^p_{\mathfrak{m}}(K[S])) \leqslant c(\alpha - 1) - d$ for all $p \geqslant 0$. By the definition of Castelnuovo–Mumford regularity we then get $\operatorname{reg} K[S] \leqslant c(\alpha - 1)$.

(ii) In order to prove the second statement, we choose a representation in the right side of (3) with the smallest sum of the coefficients $\sum n_l$. By the definition of the reduction number we must have $\sum n_l \leqslant r := r(S)$. Hence from (3) we get

$$x_{[i]} + \alpha \leqslant \sum_{l=1}^{c} n_l a_{l[i]} \leqslant (\alpha - 1) \sum_{l=1}^{c} n_l \leqslant r(\alpha - 1),$$

which yields

$$\sum_{i=1}^{d} x_{[i]} + d\alpha \leqslant dr(\alpha - 1).$$

That means $\deg x \leqslant dr - \lceil \frac{dr}{a} \rceil - d$, which implies $\operatorname{reg} K[S] \leqslant dr - \lceil \frac{dr}{a} \rceil$. □

Remark 3.3.

(i) Let

$$a^*(S) = \max\{a(H^i_{\mathfrak{m}}(K[S]));\ i \geqslant 0\}.$$

This is an interesting invariant and recently studied in several papers (see, e.g., [12] and [19]). In the above theorem we have in fact proved that

$$a^*(S) \leqslant \min\{c(\alpha - 1) - d,\ dr - \lceil \tfrac{dr}{\alpha} \rceil - d\}.$$

(ii) Since we already know that $d + a(H^d_{\mathfrak{m}}(K[S])) \leqslant \deg K[S] - c$, any bound on $a^*(S)$ will provide a bound on $\operatorname{reg} K[S]$ according to the following formula:

$$\operatorname{reg} K[S] \leqslant \max\{a^*(S) + d - 1,\ \deg K[S] - c\}.$$

Lemma 3.4. $\alpha \mid \deg K[S]$. *In particular* $\alpha \leqslant \deg K[S]$.

Proof. Denote by $\operatorname{vol}(x_1, \dots, x_d)$ the Euclidean volume of the convex polytope spanned by the points $0, x_1, \dots, x_d$ in the affine space $\mathbf{Q}^d$. From the relation

$$\frac{a_{j[1]}}{\alpha} e_1 + \cdots + \frac{a_{j[d]}}{\alpha} e_d = a_j,$$

we see that

$$\frac{a_{j[i]}}{\alpha} = \left| \frac{\det(a_j, e_1, \dots, \widehat{e_i}, \dots, e_d)}{\det(e_1, \dots, e_d)} \right| = \frac{\operatorname{vol}(a_j, e_1, \dots, \widehat{e_i}, \dots, e_d)}{\operatorname{vol}(e_1, \dots, e_d)}.$$

Fix a basis $b_1, \dots, b_d$ of the group $\mathbf{Z}(\mathcal{A})$ generated by $\mathcal{A}$. It is known that (see, e.g., [15, Theorem 4.16]):

$$\deg K(S) = \frac{\operatorname{vol}(e_1, \dots, e_d)}{\operatorname{vol}(b_1, \dots, b_d)}.$$

Moreover $\operatorname{vol}(a_j, e_1, \dots, \widehat{e_i}, \dots, e_d) = n_{ji} \operatorname{vol}(b_1, \dots, b_d)$ for some nonnegative integers n_{ji}. Hence $\deg K[S] a_{j[i]} = n_{ji}\alpha$ for all j, i. Since $a_{j[i]}$, $j = 1, \dots, c$, $i = 1, \dots, d$, are relatively prime, it follows that $\deg K[S]$ is divisible by α. □

The best bound known until now for toric graded rings is due to Sturmfels who showed that $\operatorname{reg} K[S] \leqslant c(c+d) \deg K[S] - 1$ (see [16, Theorem 4.5]). Moreover we may restrict ourself to the case $\deg K[S] \geqslant \operatorname{codim} K[S] + 2$, since it is well-known that $\operatorname{reg} K[S] \leqslant 1$ if $\deg K[S] = \operatorname{codim} K[S] + 1$ (see [3] or [5]). For simplicial semigroups S our main result below gives bounds which are not too far from the one given in Eisenbud–Goto's Conjecture.

Theorem 3.5. *Assume that* $\deg K[S] \geqslant \operatorname{codim} K[S] + 2$. *The following hold*:

(i) $\operatorname{reg} K[S] \leqslant \operatorname{codim} K[S](\deg K[S] - 1)$,
(ii) $\operatorname{reg} K[S] \leqslant d(\deg K[S] - \operatorname{codim} K[S] - 2) + 2$.

Proof. The first inequality follows from Lemma 3.4 and Theorem 3.2(i). If $r(S) \leqslant 1$ then, in view of Theorem 3.1(i), the statement (ii) is trivially true. If $r(S) \geqslant 2$ then (ii) follows from Theorem 3.1(ii) and Theorem 1.1. □

For $d = 2$ we always have $\alpha = \deg K[S]$ in Lemma 3.4. An easy example with $e_1 = (d, 0, \ldots, 0), \ \ldots, \ e_d = (0, \ldots, 0, d)$ and $a_1 = (1, \ldots, 1)$ shows that $\alpha = \deg K[S]$ can happen in any dimension. However in many cases $\alpha \ll \deg K[S]$ and the bound in Theorem 3.2(i) is much better than the one given in Eisenbud–Goto's Conjecture if $\operatorname{codim} K[S]$ is not too big. The following result is an immediate consequence of Theorem 3.2(i):

Corollary 3.6. *Assume that* $\operatorname{codim} K[S] \leqslant \deg K[S]/\alpha$. *Then* $\operatorname{reg} K[S] \leqslant \deg K[S] - \operatorname{codim} K[S]$.

Consider again Example 1.4: Let $\mathcal{A}$ contain the following d elements $(u_{[1]}, \ldots, u_{[d]})$, $(u_{[1]} - 1, u_{[2]} + 1, u_{[3]}, \ldots, u_{[d]}), \ldots, (u_{[1]} - 1, u_{[2]}, \ldots, u_{[d-2]}, u_{[d-1]} + 1, u_{[d]})$, where $u_{[1]}, \ldots, u_{[d]}$ are nonnegative integers such that $u_{[1]} + \cdots + u_{[d]} = \alpha$ and $0 < u_{[1]} < d$. Then Eisenbud–Goto's conjecture holds if $c = \operatorname{codim} K[S] \leqslant \alpha^{d-2}$. Moreover, if $c \ll \alpha^{d-2}$ then the bound $c(\alpha - 1)$ is much less than $\deg K[S] - \operatorname{codim} K[S]$. We can derive other examples using Lemmata 1.2, 1.3 and Theorem 3.1. Although the set of such affine simplicial semigroup rings is rather big, we still need a restriction on the codimension. From this point of view the following partial result is of interest. With its proof we also would like to show that for a concrete given affine simplicial semigroup, a deeper analysis of the proof of Theorem 3.2 could lead to a better result.

Proposition 3.7. *Assume that* $\deg K[S] = \alpha^{d-1}$ *and* $\alpha \leqslant d - 1$. *Then Eisenbud–Goto's conjecture holds, i.e.,*

$$\operatorname{reg} K[S] \leqslant \deg K[S] - \operatorname{codim} K[S].$$

Proof. If $\mathcal{A} = M_{\alpha,d}$ then it is well-known that S is a normal semigroup, i.e., $S = \mathbf{Z}(S) \cap \mathbf{N}^d$. In this case $K[S]$ is a Cohen–Macaulay domain (see [7]). Hence by [17] the Eisenbud–Goto's conjecture holds. Thus we may assume that $\mathcal{A} \subset M_{\alpha,d}$.

Let $n = \sharp\mathcal{A}$. Then we have

$$n \leqslant \sharp M_{\alpha,d} - 1 = \binom{\alpha + d - 1}{d - 1} - 1. \tag{4}$$

By Remark 3.3 it suffices to show that

$$(n - d)(\alpha - 1) - 1 \leqslant \alpha^{d-1} - (n - d),$$

which is equivalent to

$$n \leqslant \alpha^{d-2} + d. \tag{5}$$

The rest of the proof will be done in the following four claims. □

The cases $\alpha \geqslant 3$, $d \geqslant 6$ and $\alpha = 4$, $d = 5$ follow from the above inequalities (4), (5) and the following claim.

Claim 1. Assume that one of the following conditions holds:

(i) $3 \leqslant \alpha \leqslant d - 1$ and $d \geqslant 6$,
(ii) $\alpha = 4$ and $d = 5$.

Then

$$\frac{(\alpha + 1)\cdots(\alpha + d - 1)}{(d - 1)!} \leqslant \alpha^{d-2} + d + 1. \tag{6}$$

Proof. The case (ii) and the case $d = 6$, $3 \leqslant \alpha \leqslant 5$ can be checked directly. Let $\alpha \geqslant 3$ and $d \geqslant 7$. Note that for $i \geqslant 3$ we have $1/i + 1/\alpha \leqslant 2/3$. Hence $(\alpha + i)/i \leqslant 2\alpha/3$. The left-hand side of (6) is

$$\leqslant \frac{(\alpha + 1)(\alpha + 2)}{2}\left(\frac{2}{3}\alpha\right)^{d-3} = \frac{1}{2}\left(\frac{2}{3}\right)^{d-3}\alpha^{d-3}(\alpha^2 + 3\alpha + 2).$$

Hence it suffices to show that

$$\frac{1}{2}\left(\frac{2}{3}\right)^{d-3}\alpha^{d-3}\bigl(\alpha^2 + 3\alpha + 2\bigr) \leqslant \alpha^{d-2},$$

which is equivalent to

$$2\left(\frac{2}{3}\right)^{d-3}\alpha \geqslant \alpha^2 + 3\alpha + 2. \tag{7}$$

Since $d \geqslant 7$, we have

$$\begin{aligned}\left(\frac{3}{2}\right)^{d-3} &= \left(1+\frac{1}{2}\right)^{d-3} > 1+\frac{d-3}{2}+\frac{(d-3)(d-4)}{2.4}+\frac{(d-3)(d-4)(d-5)}{2.3.8}\\ &\geqslant 1+\frac{d-3}{2}+\frac{d-4}{2}+\frac{1}{2}=d-2.\end{aligned}$$

Since $\alpha < d-1$, it is easy to check that $2(d-2)\alpha > \alpha^2+3\alpha+2$. Therefore (7) holds, as required. □

Claim 2. Eisenbud–Goto's conjecture holds for $\alpha = 2$.

Proof. We have $\sharp M_{2,d} = d(d+1)/2$. If $n \leqslant d(d+1)/2-2$ then (5) is satisfied. By (4), the only case to be considered is $n = d(d+1)/2-1$. We will show that in this case

$$a^*(S) \leqslant 0.$$

We can assume that $\mathcal{A} = M_{2,d} \setminus \{b = (1,1,0,\dots,0)\}$. Rewrite the main relation (3) in the proof of Theorem 3.2 as follows:

$$x + e_i + u_I = \sum_{h\neq i} m_h e_h + \sum_{a_j\notin\mathcal{P}_i} n_j a_j + \sum_{a_l\in\mathcal{P}_i} p_l a_l.$$

In this case $n_j = 0, 1$. Note that for any $b_1, b_2 \notin \mathcal{P}_i$ such that the 2-dimensional face of $\mathcal{P}$ containing these elements does not contain b, then we have $b_1 + b_2 \in e_i + \mathcal{A}$ (see Fig. 1).

So these two elements cannot simultaneously appear in the right side of the above relation. Since there is at most one 2-dimensional face of $\mathcal{P}$ containing two points from $\mathcal{A} \setminus \mathcal{P}_i$ and b, it even implies that in the above relation we must have $\sum n_j \leqslant 2$. Hence $x_{[i]} + 2 \leqslant 2$, i.e., $x_{[i]} \leqslant 0$ for all i. So we have $\deg x \leqslant 0$, and $a^*(S) \leqslant 0$. Since

$$\deg K[S] - c = 2^{d-1} + d + 1 - \frac{d(d+1)}{2} \geqslant d-1 \geqslant a^*(S) + d - 1,$$

the claim follows by Remark 3.3 □

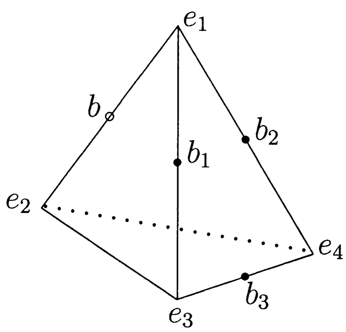

Fig. 1.

Claim 3. Eisenbud–Goto's conjecture holds for $\alpha = 3$ and $d = 4$.

Proof. We have $\sharp M_{3,4} = 20$. If $n \leqslant 13$ the inequality (5) is satisfied. By (4), we have only to consider the case $14 \leqslant n \leqslant 19$. In this case $\mathcal{A}$ is obtained from $M_{3,4}$ by deleting 1 to 6 points. We divide points of $M_{3,4} \setminus \{e_1, \dots, e_4\}$ into two types:

Type 1: points which are lying in an edge of $\mathcal{P}$,
Type 2: points which are lying inside a 2-dimension face of $\mathcal{P}$.

Consider two cases:

Case 1: All points of Type 2 belong to $\mathcal{A}$. There are 4 such points and we enumerate them as $a_1, \dots, a_4$. We rewrite the relation (3) as follows:

$$x + e_i + u_I = \sum_{h \neq i} m_h e_h + \sum_{j=1}^{4} n_j a_j + \sum_{l=5}^{c} p_l a_l. \tag{8}$$

Look at a face of $\mathcal{P}$, say $\overline{\mathcal{P}}_4$ (see Fig. 2).

$b_1, \dots, b_6$ are points of Type 1 and b is a point of Type 2. By the assumption $b \in \mathcal{A}$. Using relations of the following types: $b_1 + b_3 = e_1 + b$, $b_1 + b_6 = 2b$, $b_1 + b_2 = e_1 + e_2$, $b_1 + b_4 + b_5 = 3b$, $2b_1 + b_4 = e_1 + 2b$ to replace elements of Type 1 in the relation (8) by $e_1, \dots, e_4$ and elements of Type 2 which already appeared there, one can assume that in each facet of $\mathcal{P}$ there are at most two elements of Type 1 (counted with multiplicity) appeared in (8). Hence, if some element $a_l \notin \mathcal{P}_i$ $(l \geqslant 5)$ appears in the right side with multiplicity 2 (i.e., $p_l = 2$), then all other points of Type 1 in (8) must lie on $\mathcal{P}_i$. Moreover from relations of the type $b_1 + b_3 = e_1 + b$ it follows that if there is already an element $a_l \notin \mathcal{P}_i$, $l \geqslant 5$, with $p_l = 1$ and $a_{l[i]} = 2$, then there are at most two other points of Type 1 which appear in (8) and do not belong to $\mathcal{P}_i$, and the ith coordinates of these elements are 1. All these together imply that

$$\sum_{l=5}^{c} p_l a_{l[i]} \leqslant 4.$$

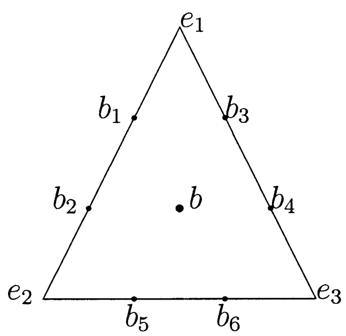

Fig. 2.

Since $n_j \leqslant 2$, from (8) we get

$$x_{[i]} + 3 \leqslant 4 + 2\sum_{j=1}^{4} a_{j[i]}.$$

Summing up all $x_{[i]}$ we then get $\deg x \leqslant 9$, and so $a^*(S) \leqslant 9$. Since $\deg K[S] - c = 27 - (n-4) \geqslant 12 \geqslant a^*(S) + 3$, by Remark 3.3 it follows that $\operatorname{reg} K[S] \leqslant \deg K[S] - c$.

Case 2: At least one point of Type 2 does not belong to $\mathcal{A}$. Then $\mathcal{P}$ has $k \geqslant c - 9 \geqslant 1$ full edges. Look at one full edge, say $\overline{e_1 e_2}$ (see Fig. 2). We have $2b_1 = e_1 + b_2$, $2b_2 = e_2 + b_1$ and $b_1 + b_2 = e_1 + e_2$. Hence one can assume that at most one point of Type 1 in a full edge can appear in the relation

$$x + e_i + u_I = \sum_{h \neq i} m_h e_h + \sum_{a_j \notin \mathcal{P}_i} n_j a_j + \sum_{a_l \in \mathcal{P}_i} p_l a_l, \tag{9}$$

and its coefficient is 1. Let ε_i denote the number of full edges passing through e_i. Denote by F the set of points of Type 1 in full edges and by NF the set of remaining points of Type 1. From the above relation we then get

$$\begin{aligned} x_{[i]} + 3 &\leqslant \sum_{a_j \in F \setminus \mathcal{P}_i} n_j a_{j[i]} + \sum_{a_j \in NF \setminus \mathcal{P}_i} n_j a_{j[i]} \\ &\leqslant \sum_{a_j \in F \setminus \mathcal{P}_i} a_{j[i]} - \varepsilon_i + 2 \sum_{a_j \in NF \setminus \mathcal{P}_i} a_{j[i]} \\ &= \sum_{a_j \in F} a_{j[i]} - \varepsilon_i + 2 \sum_{a_j \in NF} a_{j[i]}. \end{aligned}$$

Since $\mathcal{P}$ has k full edges, $\sharp F = 2k$ and $\varepsilon_1 + \cdots + \varepsilon_4 = 2k$. Taking the sum over all i we get

$$x_1 + \cdots + x_4 + 12 \leqslant 2k.3 - 2k + 2(c - 2k).3 = 6c - 8k.$$

Hence

$$a^*(S) \leqslant 2c - 2k - 4 - \left\lceil \frac{2k}{3} \right\rceil.$$

By Remark 3.3 we have

$$\operatorname{reg} K[S] \leqslant \max\left\{2c - 2k - 1 - \left\lceil \frac{2k}{3} \right\rceil,\ \deg K[S] - c\right\}. \tag{10}$$

Looking at Table 1 we can conclude from (10) that we have only to work with the case $c = 15$. In this case we may assume that $\mathcal{A} = M_{3,4} \setminus \{(1,1,1,0)\}$. Then one can even assume that each of the facets $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ has at most one point of Type 1 appeared

Table 1

c	k	$2c-2k-1-\lceil\frac{2k}{3}\rceil$	$\deg K[S]-c$
10	$\geqslant 1$	$\leqslant 16$	17
11	$\geqslant 2$	$\leqslant 15$	16
12	$\geqslant 3$	$\leqslant 15$	15
13	$\geqslant 4$	$\leqslant 14$	14
14	$\geqslant 5$	$\leqslant 13$	13
15	6	< 13	12

in (9). From that one can conclude that $\operatorname{reg} K[S] \leqslant 10 < \deg K[S] - c = 12$. (In fact in this case, using [20], Theorem 3.3 and Corollary 3.4 one can show that $H^1_{\mathfrak{m}}(K[S]) \cong K(-1)$, $H^2_{\mathfrak{m}}(K[S]) = H^3_{\mathfrak{m}}(K[S]) = 0$ and $a(H^4_{\mathfrak{m}}(K[S])) = -2$. Hence $\operatorname{reg} K[S] = 2$.) $\square$

Claim 4. Eisenbud–Goto's conjecture holds for $\alpha = 3$ and $d = 5$.

Proof. We have $\sharp M_{3,5} = 30$. From (4) and (5) we see that we have only to consider two cases $n = 28, 29$, i.e., $c = 23, 24$. In this case the formula (10) also holds. Note that if $c = 23$ then $k \geqslant 8$ and if $c = 24$ then $k \geqslant 9$. From this it is immediate to see that $\operatorname{reg} K[S] < \deg K[S] - c$. $\square$

Corollary 3.8. *Assume that* $\deg K[S] = \alpha^{d-1}$ *and* $\mathcal{P}$ *has a full edge. Then* $\operatorname{reg} K[S] \leqslant \deg K[S] - 1$.

Proof. If $\alpha \leqslant d-1$ this was proved in Proposition 3.7. If $\alpha \geqslant d$, then by Theorem 3.1 and Lemma 1.2 we have $\operatorname{reg} K[S] \leqslant dr \leqslant \alpha(\alpha^{d-2} - 1) < \alpha^{d-1} - 1 = \deg K[S] - 1$. $\square$

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